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Massive U(1)s and Heterotic Five-Branes on K3

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Abstract

We systematically consider heterotic $SO(32)$ and $E_8 \times E_8$ compactifications on $K3$ with Abelian and non-Abelian backgrounds as well as an arbitrary number of five-branes. The masses of the $U(1)$ factors depend on the first Chern classes of the bundles and some combinatorial factors specifying the embedding in $SO(32)$ or E_8. The form of the generalised Green-Schwarz counter-terms in six dimensions constrains the possible heterotic five-brane actions.

Some supersymmetric examples on $K3$ realisations as toric complete intersection spaces with up to three explicit two-forms are given.

Contents

1 Introduction

Recently, it was shown that not only in four-dimensional Type II string compactifications, e.g. with intersecting or magnetised branes (for a recent review see [1]), but also in heterotic compactifications, multiple anomalous $U(1)$ factors can arise [2–5].[1]

In six-dimensional heterotic compactifications this phenomenon has been known for a long time [9, 10], and the $E_8 \times E_8$ case has been investigated in some detail motivated by F-theory, see e.g. [11–13]. However, a fully quantitative treatment of the Green-Schwarz counter-terms including non-perturbative effects is still missing. In [14], the role of additional tensor multiplets for the generalised Green-Schwarz mechanism was advertised, which is relevant for heterotic five (H5)-branes in $E_8 \times E_8$ compactifications. On the other hand, six-dimensional $SO(32)$ heterotic compactifications with multiple $U(1)$s and H5-banes have only been poorly investigated.[2]

[1] Further four-dimensional heterotic compactifications with $U(N)$ bundles can be found in [6–8].

[2] Early six dimensional heterotic string spectra with H5-branes - not discussing the detailed contributions of these to the anomaly cancellation - and purely non-Abelian gauge groups are given in [15–18], for more references see also the review [19].

In this article, we systematically study six-dimensional $SO(32)$ and $E_8 \times E_8$ heterotic string compactifications on $K3$ with an arbitrary number of H5-branes extended along the non-compact directions and multiple $U(1)$ gauge factors. We show that the masses of the Abelian gauge factors depend on the first Chern classes of the bundles and some combinatorial factors associated to the embedding of the $U(1)$s in $SO(32)$ or E_8, while the instanton numbers, i.e. second Chern characters, enter the tadpole cancellation condition. Moreover, we explicitly compute the H5-brane contributions to the generalised Green-Schwarz counter-terms and show that they do not contribute to the Abelian mass terms. We hope that the present work can help to clarify the F-theory lift of multiple heterotic $U(1)$ factors.

The paper is organised as follows: After some general remarks on $\mathcal{N} = 1$ heterotic compactifiactions on $K3$ and anomaly cancellation in six dimensions in section 2, the $SO(32)$ case is treated in full generality in section 3, and two examples with non-trivial unitary bundles are given. The generic $E_8 \times E_8$ case is treated in section 4, a class of embeddings with $U(n) \times U(m)$ backgrounds is specified and three examples are discussed exhibiting the relations among first Chern classes and massive $U(1)$ factors. Finally, the conclusions are given in section 5.

2 Six-dimensional heterotic compactifications

2.1 Some facts in six dimensions

In this section, some facts about $\mathcal{N} = 1$ heterotic compactifications on $K3$ are collected.

- The $\mathcal{N} = 1$ supersymmetric multiplets in six dimensions are the hyper, vector, tensor and supergravity multiplets with field content given in table 1.

Multiplet	Content
SUGRA	$(g_{\mu\nu}, B^+_{\mu\nu}, \psi^-_\mu)$
Tensor	$(B^-_{\mu\nu}, \phi, \chi^+)$
Vector	(A_μ, λ^-)
Hyper	$(4\varphi, \psi^+)$

Table 1: Bosonic and fermionic content of the $\mathcal{N} = 1$ multiplets in six dimensions. The index -(+) denotes a spinor of negative (positive) chirality or an (anti)selfdual two-form. Half-hyper multiplets can occur if they transform under some real representation of the gauge group.

- For $\mathcal{N}=1$ heterotic string compactifications, the net number of chiral states transforming under some bundle V on a Calabi-Yau n-fold is given by the Riemann-Roch-Hirzebruch theorem [20]

$$\chi(V)=\int_{CY_n}\mathrm{ch}(V)\mathrm{Td}(CY_n) \tag{1}$$

associated to the cohomology classes $H^*(CY_n,V)$. The non-trivial Todd classes on $K3$ are given by

$$\mathrm{Td}_0(K3)=1,\quad \mathrm{Td}_2(K3)=\frac{1}{12}c_2(K3)=2,$$

leading to

$$\chi_{K3}(V)=\mathrm{ch}_2(V)+2r \tag{2}$$

with $r=\mathrm{rank}(V)$. Since the vector and hyper multiplets in six dimensions have opposite chirality, with the sign convention in (1), the index counts

$$\chi_{K3}(V)=\#\mathrm{Vector}-\#\mathrm{Hyper}$$

multiplets in the representation associated to the bundle V. The gauge group in string compactifications is known by constructions. Thus, in contrast to four-dimensional compactifications, the complete massless spectrum can be computed from the index (2). [3]

- In general, several types of six-dimensional field theory anomalies occur which can be encoded in the well known anomaly eight-form

$$\begin{aligned}I_8 &= \frac{n_H-n_V+29n_T-273}{360}\mathrm{tr}R^4+\frac{n_H-n_V-7n_T+51}{288}(\mathrm{tr}R^2)^2 \\ &+ \frac{1}{6}\mathrm{tr}R^2\sum_A C_A\mathrm{tr}F_A^2-\frac{2}{3}\sum_A A_A\mathrm{tr}F_A^4-\frac{2}{3}\sum_A B_A(\mathrm{tr}F_A^2)^2 \\ &+ 4\sum_{A<B}C_{AB}\mathrm{tr}F_A^2\mathrm{tr}F_B^2+\frac{8}{3}\sum_{A,B}D_{AB}\mathrm{tr}F_A\mathrm{tr}F_B^3,\end{aligned} \tag{3}$$

[3]To be more precise, the bosonic index (1) is by supersymmetry equal to the fermionic Atiyah-Singer index $\int_{CY_n}\mathrm{ch}(V)\hat{A}(CY_n)$. The fermionic content in table 1 consist of a pair of symplectic Majorana spinors per multiplet. A hyper multiplet contains a complex scalar and symplectic Majorana fermion in the *complex* representation $\mathbf{R}$ as well as the CPT conjugate states in $\overline{\mathbf{R}}$, i.e. *one* hyper multiplet is denoted by $\mathbf{R}+c.c.$ and the number of such multiplets is counted by the index $\chi(V)$ of the associated bundle, with $\chi(V)=\chi(V^*)$ on $K3$. For a *pseudo-real* representation, a symplectic Majorana spinor is CPT invariant and half-hyper multiplets can occur which are taken into account in this article by allowing for half integer numbers in table 2 .

where A_A, B_A, C_A, C_{AB} and D_{AB} are coefficients encoding the number of fermions transforming under some gauge group(s), the sign given by their chirality and multiplicities from various representations taken into account. The traces formally run also over Abelian gauge factors. The complete list of coefficients including multiple Abelian factors explicitly can be found, e.g., in [21].

As we will show in section 2.3, factorizable anomalies can be cancelled by Green-Schwarz counter-terms whereas the $\mathrm{tr}R^4$ and the non-Abelian $\mathrm{tr}F^4$ anomalies have to be absent for a consistent six-dimensional massless spectrum.

- The supersymmetry condition, the so called Donaldson-Uhlenbeck-Yau equation (DUY), on some *holomorphic* background gauge field strength $\overline{F}$ in six dimensions is given by

$$\int_{K3} J \wedge \overline{F} = 0, \tag{4}$$

where J is the Kähler form on $K3$. Potential loop corrections with the same parity symmetry under $\overline{F} \to -\overline{F}$ would involve $\overline{F}^{2n+1}$ and must be absent for dimensional reasons. It is thus expected that (4) is perturbatively exact. Further support for this conjecture stems from the fact that the ten-dimensional dilaton forms the scalar degree of freedom of the six-dimensional universal tensor multiplet. As we will show, this tensor multiplet contributes to the generalised Green-Schwarz mechanism, but does not acquire a mass. Any loop correction to (4) would be in contradiction to this observation.

- The Bianchi identity on the three-form field strength results in the so called tadpole cancellation condition given by

$$\mathrm{tr}\overline{F}^2 - \mathrm{tr}\overline{R}^2 - 16\pi^2 N_{H5} = 0 \tag{5}$$

in cohomology on $K3$, where N_{H5} is the total number of H5-branes for both $SO(32)$ and $E_8 \times E_8$ string compactifications.

- A gauge field $F = dA$ in D dimensions becomes massive through a coupling [22, 23]

$$m \int_{\mathbb{R}^{1,D-1}} \gamma^{(D-2)} \wedge F \sim m \int_{\mathbb{R}^{1,D-1}} \left(\star_D d\beta^{(0)}\right) \wedge A, \tag{6}$$

with the duality relation $d\gamma^{(D-2)} \sim \star_D d\beta^{(0)}$ among the $(D-2)$ form γ and scalar β. The coupling on the left hand side naturally arises in the dimensional field theory reduction presented in section 2.3 while the right

hand side has the more familiar shape [9].
The massive gauge factor remains as a perturbative global symmetry in the Lagrangian.
Heterotic compactifications contain at most 16 massive $U(1)$ factors from the perturbative gauge group.

- Consistent models are further constrained by K-theory. In [24, 25] it has been shown that at least for compactifications to four dimensions, the K-theory constraint is given by

$$c_1(V_{total}) = 0 \bmod 2, \tag{7}$$

where V_{total} is the total background gauge bundle. Since K-theory is associated to $\mathbb{Z}_2$ valued charges, we expect the condition (7) to hold also for $K3$ compactifications.

- If supersymmetry is preserved, a vector multiplet becomes massive by absorbing a complete hyper multiplet. The coupling (6) absorbs one scalar descending from the ten-dimensional antisymmetric tensor and the DUY equation (4) freezes one geometric modulus. The freezing of the remaining two scalars in a hyper multiplet is best seen in the $\mathcal{N} = 2$, $d = 4$ language: a hyper multiplet contains a scalar triplet of $SU(2)_R$, for which three Fayet-Iliopoulos terms arise, see e.g. [22]. In order to preserve supersymmetry, all three D-terms have to vanish simultaneously. In the present case, the triplet consists of geometric moduli, and one D-term condition provides the DUY equation (4), while the other two supersymmetry conditions are encoded in the requirement of a holomorphic vector bundle. The two geometric moduli which would deform the vector bundle from a pure $(1,1)$-form to contain a $(2,0)$ or $(0,2)$ piece are frozen.

2.2 K3 toy models as complete intersection spaces

There exist three simple possibilities to express $K3$ as complete intersections of projective spaces without introducing singularities. The three possibilities including the Quartic are as follows,

$$\mathcal{M}_1 = \mathbb{P}_3[4], \qquad \mathcal{M}_2 = \begin{array}{c} \mathbb{P}_1 \\ \mathbb{P}_2 \end{array} \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \qquad \mathcal{M}_3 = \begin{array}{c} \mathbb{P}_1 \\ \mathbb{P}_1 \\ \mathbb{P}_1 \end{array} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix},$$

with in the toric description up to three explicit two-forms.

Let η_i denote the $(1,1)$-forms on the up to three projective factors. The Stanley-Reisner ideals are then given by

$$SR^{(1)} = \{\eta^4\}, \qquad SR^{(2)} = \{\eta_1^2, \eta_2^3\}, \qquad SR^{(3)} = \{\eta_1^2, \eta_2^2, \eta_3^2\},$$

leading to the intersection forms

$$I_2^{(1)} = 4\eta^2, \qquad I_2^{(2)} = 3\eta_1\eta_2 + 2\eta_2^2, \qquad I_2^{(3)} = 2\eta_1\eta_2 + 2\eta_1\eta_3 + 2\eta_2\eta_3. \tag{8}$$

It can be checked explicitly that all three manifolds have complex dimension two and

$$c_1(T\mathcal{M}_i) = 0, \qquad c_2(T\mathcal{M}_i) = 24, \quad i = 1, 2, 3. \tag{9}$$

The conditions for the parameter space of a model to lie inside the Kähler cone are as follows,

$$\int_{K3} J \wedge \eta_i \stackrel{!}{>} 0 \quad \text{for all } i, \qquad \int_{K3} J \wedge J \stackrel{!}{>} 0, \tag{10}$$

where J is the Kähler form on $\mathcal{M}_i$.[4] Expanding the Kähler form in terms of the $(1,1)$-forms ($\ell_s \equiv 2\pi\sqrt{\alpha'}$)

$$J = \ell_s^2 \sum_i \rho_i \eta_i$$

gives the conditions for the three $K3$ realisations presented here

$$\begin{aligned} &(1) \quad \rho \stackrel{!}{>} 0, \\ &(2) \quad \rho_2 \stackrel{!}{>} 0, \qquad 3\rho_1 + \rho_2 \stackrel{!}{>} 0, \\ &(3) \quad \rho_i + \rho_j \stackrel{!}{>} 0 \text{ for } i \neq j, \qquad \rho_1\rho_2 + \rho_1\rho_3 + \rho_2\rho_3 \stackrel{!}{>} 0. \end{aligned}$$

Abelian bundles are specified completely by their first Chern classes, $c_1(L) = \sum_i q_i \eta_i$, and the DUY equations for the different cases are computed using the corresponding intersection forms (8),

$$\begin{aligned} &(1) \quad q\rho \stackrel{!}{=} 0, \\ &(2) \quad 3q_1\rho_2 + 3q_2\rho_1 + 2q_2\rho_2 \stackrel{!}{=} 0, \\ &(3) \quad q_1[\rho_2 + \rho_3] + q_2[\rho_1 + \rho_3] + q_3[\rho_1 + \rho_2] \stackrel{!}{=} 0. \end{aligned} \tag{11}$$

If $K3$ is realised as the Quartic $\mathcal{M}_1 = \mathbb{P}_3[4]$, non-trivial line bundles cannot solve the DUY equation, and only models with pure $SU(n)$ bundles can preserve supersymmetry. Moreover, line bundles are specified (up to a sign) by their second Chern characters due to $\mathrm{ch}_2(L) = 2q^2$. The situation is different for the two remaining $K3$ realisations $\mathcal{M}_2$ and $\mathcal{M}_3$, as we will show in some examples in sections 3.3, 3.4 and 4.4.

[4] In the formulation of the full $(3, 19)$ lattice, these conditions are replaced by the Kähler form being selfdual.

A vector bundle V of rank r on a complete intersection manifold $\mathcal{M}$ can be defined by the cohomology of the monad as $V = \mathrm{Ker}(f)/\mathrm{Im}(g)$,

$$0 \to \mathcal{O}|_{\mathcal{M}}^{\oplus p} \xrightarrow{g} \oplus_{a=1}^{r+p+q} \mathcal{O}(n_1^a, \ldots, n_k^a)|_{\mathcal{M}} \xrightarrow{f} \oplus_{b=1}^{q} \mathcal{O}(m_1^b, \ldots, m_k^b)|_{\mathcal{M}} \to 0,$$

with $p \geq 0, q \geq 1$ and k the number of $(1,1)$-forms on $\mathcal{M}$ (for more details on the notation see [2]), or alternatively via the exact sequence

$$0 \to V \to \oplus_{a=1}^{r+q} \mathcal{O}(n_1^a, \ldots, n_k^a)|_{\mathcal{M}} \xrightarrow{f} \oplus_{b=1}^{q} \mathcal{O}(m_1^b, \ldots, m_k^b)|_{\mathcal{M}} \to 0.$$

In both cases, the Chern classes are computed from

$$c(V) = \frac{\prod_a \left(1 + \sum_i n_i^a \eta_i\right)}{\prod_b \left(1 + \sum_i m_i^b \eta_i\right)},$$

leading in particular to

$$\begin{aligned} c_1(V) &= \sum_i \left(\sum_a n_i^a - \sum_b m_i^b \right) \eta_i, \\ \mathrm{ch}_2(V) &= \frac{1}{2} \sum_i \left(\sum_a (n_i^a)^2 - \sum_b (m_i^b)^2 \right) \eta_i^2 + \sum_{i<j} \left(\sum_a n_i^a n_i^a - \sum_b m_i^b m_j^b \right) \eta_i \eta_j. \end{aligned}$$

A necessary condition for a well-defined stable bundle V is $n_i^a, m_i^b, m_i^b - n_i^a \geq 0$ for all i, a, b and $(m_1^b - n_1^a, \ldots, m_k^b - n_k^a) \neq (0, \ldots, 0)$. Stability is guaranteed if all defining maps f and g have maximal rank. We will, however, not check the latter condition explicitly for the examples given in this article.

The DUY equations for a vector bundle V are given by (11) when replacing $q_i \to \left(\sum_a n_i^a - \sum_b m_i^b\right)$.

2.3 The perturbative Green-Schwarz counter-terms

If some general bundle is embedded into $SO(32)$ or $E_8 \times E_8$, several types of anomalies involving gauge and gravitational fields can occur: while $\mathrm{tr}R^4$ and $\mathrm{tr}F^4$ field theory anomalies in six dimensions have to be absent, factorizable anomalies can in general be cancelled by a generalized Green-Schwarz mechanism [9,26]. In contrast to four dimensions, where only mixed and pure Abelian anomalies are compensated, in six dimensions also anomalies involving only gravity and non-Abelian gauge fields have counter-terms.

In the following, we perform the dimensional reduction of the heterotic string on $K3$ similar to the reduction on Calabi-Yau three-folds treated in [2,3]. The relevant couplings linear in the antisymmetric tensor $B^{(2)}$ and its ten-dimensional

dual $B^{(6)}$ arise from the kinetic and the one-loop counter-term in ten dimensions,[5]

$$\begin{aligned} S_{kin} &= -\frac{\pi}{\ell_s^8}\int_{\mathbb{R}^{1,9}} e^{-2\phi_{10}}\, H\wedge\star_{10} H, \\ S_{1-loop} &= \frac{1}{24\,(2\pi)^3\,\ell_s^2}\int_{\mathbb{R}^{1,9}} B^{(2)}\wedge X_8, \end{aligned} \tag{12}$$

with the field strength $H^{(3)} = dB^{(2)} - \frac{\alpha'}{4}(\omega_Y - \omega_L)$. The anomaly eight-form is given by [26]

$$X_8 = \frac{1}{24}\mathrm{Tr}F^4 - \frac{1}{7200}\left(\mathrm{Tr}F^2\right)^2 - \frac{1}{240}\left(\mathrm{Tr}F^2\right)\left(\mathrm{tr}R^2\right) + \frac{1}{8}\mathrm{tr}R^4 + \frac{1}{32}\left(\mathrm{tr}R^2\right)^2 \tag{13}$$

for both the $SO(32)$ and $E_8\times E_8$ theories. Denoting by $X_{n+\overline{m}}$ a form with n legs along $\mathbb{R}^{1,5}$ and m legs along $K3$, the terms relevant for the Green-Schwarz mechanism take the form

$$\begin{aligned} S_{kin} &= \frac{1}{8\pi\ell_s^6}\int_{\mathbb{R}^{1,5}\times K3}\left(\left[\mathrm{tr}\overline{F}^2 - \mathrm{tr}\overline{R}^2\right]\wedge B^{(6)} + \left[\mathrm{tr}F^2 - \mathrm{tr}R^2\right]\wedge B^{(\bar{4}+2)} + 2\,\mathrm{tr}(F\overline{F})\wedge B^{(\bar{2}+4)}\right), \\ S_{1-loop} &= \frac{1}{24(2\pi)^3\ell_s^2}\int_{\mathbb{R}^{1,5}\times K3}\left(B^{(\bar{2})}\wedge X_{\bar{2}+6} + B^{(2)}\wedge X_{\bar{4}+4}\right). \end{aligned} \tag{14}$$

Expanding in terms of a basis $\{\omega_k\}_{k=0,\ldots,h_{11}+1}$ of two-forms on $K3$ as well as its dual basis $\{\widehat{\omega}_l\}_{l=0,\ldots,h_{11}+1}$, i.e. $\int_{K3}\omega_k\wedge\widehat{\omega}_l = \delta_{kl}$,[6]

$$\begin{aligned} B^{(2)} &= b_0^{(2)} + \ell_s^2\sum_{k=0}^{h_{11}+1} b_k^{(0)}\omega_k, \qquad B^{(6)} = c_0^{(6)} + \ell_s^2\sum_{k=0}^{h_{11}+1} c_k^{(4)}\widehat{\omega}_k + \ell_s^4 c_0^{(2)}\mathrm{vol}_4, \\ \bar{Y}_{\bar{2}} &= \sum_{k=0}^{h_{11}+1}\left[\bar{Y}\right]^k\omega_k = \sum_{k=0}^{h_{11}+1}\left[\bar{Y}\right]^{\widehat{k}}\widehat{\omega}_k, \end{aligned} \tag{15}$$

with $\int_{K3}\mathrm{vol}_4 = 1$ and $k = 0, h_{11}+1$ labeling the (2,0) and (0,2) form, respectively, leads to the six-dimensional couplings linear in $b_k^{(i)}$ and $c_k^{(i)}$

$$\begin{aligned} S_{kin} = &\frac{1}{8\pi\ell_s^6}\int_{\mathbb{R}^{1,5}} c_0^{(6)}\int_{K3}\left[\mathrm{tr}\overline{F}^2 - \mathrm{tr}\overline{R}^2\right] \\ &+\frac{1}{8\pi\ell_s^2}\int_{\mathbb{R}^{1,5}} c_0^{(2)}\wedge\left[\mathrm{tr}F^2 - \mathrm{tr}R^2\right] \\ &+\frac{1}{4\pi\ell_s^4}\sum_{k=0}^{h_{11}+1}\int_{\mathbb{R}^{1,5}} c_k^{(4)}\wedge\left[\mathrm{tr}(F\overline{F})\right]^k, \end{aligned}$$

[5] The prefactor of the Green-Schwarz counter-term has been derived in [31] from M-theory reduction and by S-duality.

[6] The following expansion of $\bar{Y}$ differs by a factor of 2π from the one in [2,3].

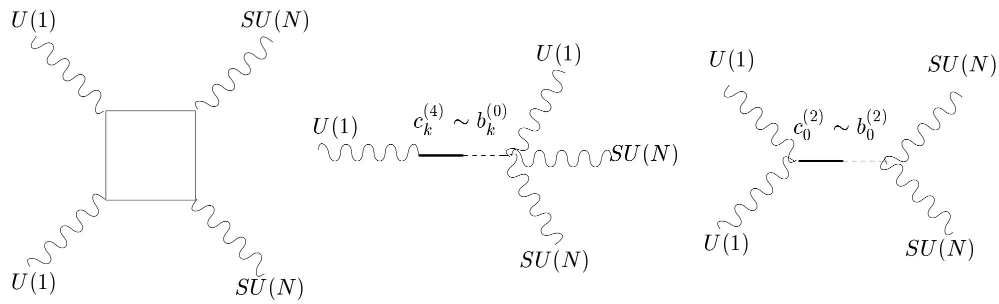

Figure 1: A six-dimensional anomalous diagram and its two possible types of Green-Schwarz counter diagrams.

$$\begin{aligned} S_{1-loop} &= \frac{1}{24(2\pi)^3} \sum_{k=0}^{h_{11}+1} \int_{\mathbb{R}^{1,5}} b_k^{(0)} \, [X_{\overline{2}+6}]^{\widehat{k}} \\ &+ \frac{1}{24(2\pi)^3 \ell_s^2} \int_{\mathbb{R}^{1,5}} b_0^{(2)} \wedge \int_{K3} X_{\overline{4}+4}. \end{aligned} \tag{16}$$

The scalars $b_k^{(0)}$ (or their dual four-forms $c_k^{(4)}$) belong in six dimensions to the 20 hyper multiplets encoding the $K3$ geometry. In more detail, 19 hyper multiplets contain one scalar $b_k^{(0)}$ pertaining to the anti-selfdual two-forms on $K3$ and three geometric moduli each, while one hyper multiplet contains the remaining three $b_k^{(0)}$ belonging to the three selfdual two-forms on $K3$ and the overall volume modulus. The two-forms $b_0^{(2)}$ (or its duals $c_0^{(2)}$) contain the selfdual tensor $B_{\mu\nu}^{+}$ of the six-dimensional supergravity multiplet and the anti-selfdual tensor $B_{\mu\nu}^{-}$ of the universal tensor multiplet. The scalar degree of freedom in the universal tensor multiplet is given by the dilaton.

The terms (16) combine to two types of Green-Schwarz counter-terms [23] depicted in figure 1,

$$\mathcal{I}_{pert} = \frac{1}{48(2\pi\ell_s)^4} \int_{K3} \left(\mathrm{tr}(F\overline{F}) \wedge X_{\overline{2}+6} + \frac{1}{2} \left(\mathrm{tr}F^2 - \mathrm{tr}R^2 \right) \wedge X_{\overline{4}+4} \right), \tag{17}$$

where the first term corresponds to the sum over all counter diagrams of the first type with $c_k^{(4)} \sim b_k^{(0)}$ exchange[7] and the last term corresponds to the second Green-Schwarz counter diagram involving $c_0^{(2)} \sim b_0^{(2)}$ couplings. As in four dimensions, massive $U(1)$ factors occur if some coupling

$$S_{mass} = \frac{1}{4\pi\ell_s^4} \sum_{k=0}^{h_{11}+1} \int_{\mathbb{R}^{1,5}} c_k^{(4)} \wedge \left[\mathrm{tr}(F\overline{F}) \right]^k \tag{18}$$

[7]These have been missed in [9].

exists, but the Abelian factor can still be anomaly-free if the (sum of the) Green-Schwarz diagrams vanish(es). On the other hand, contrarily to the four-dimensional case, $U(1)^4$ and $U(1)^2 - \text{tr}R^2$ anomalies can be cancelled by diagrams involving only $b_0^{(2)} \sim c_0^{(2)}$ exchange in the same way as $(\text{tr}_{SU(N)}F^2)^2$ and $\text{tr}_{SU(N)}F^2 - \text{tr}R^2$ counter-terms exist. An anomalous $U(1)$ factor in six dimensions can hence stay massless and a massive $U(1)$ factor can be anomaly-free. These effects are important for the correct identification of the F-theory lift.

Note that a supersymmetric background $\overline{F}$ is of type $(1,1)$ and can thus only receive a mass via couplings to $c_k^{(4)}$ for $k = 1, \dots, h_{11}$. This will be the case in the examples discussed in sections 3.3, 3.4 and 4.4.

Except from the perturbative terms (17), H5-branes contribute to the anomaly cancellation. While for $SO(32)$ compactifications, they provide symplectic gauge factors as well as charged matter, in the $E_8 \times E_8$ theory they provide one tensor and one hyper multiplet per H5-brane.

The detailed shape of the polynomials $X_{\overline{2}+6}$, $X_{\overline{4}+4}$ as well as the H5-brane contributions is discussed in sections 3 and 4 for $SO(32)$ and $E_8 \times E_8$ compactifications, respectively.

3 The $SO(32)$ case

3.1 Massless spectrum, H5-branes and tadpole cancellation

For $SO(32)$ heterotic compactifications, we decompose

$$SO(32) \supset SO(2M) \times \prod_{i=1}^{K} U(n_i\, N_i)$$

with $M + \sum_{i=1}^{K} n_i\, N_i = 16$ and take bundles

$$W = \oplus_{i=1}^{K} V_i \tag{19}$$

with structure group $G = \prod_{i=1}^{K} U(n_i)$ which leave the non-Abelian gauge group

$$H = SO(2M) \times \prod_{i=1}^{K} SU(N_i)$$

as well as the massless $U(1)$ factors to be computed below in the low energy effective field theory. This prescription generalises the embedding of $U(1)$ factors in $SO(32)$ discussed in [9] to $U(n_i)$ bundles with $n_i \geq 1$.

N_a H5-branes extended along the non-compact dimensions and at the same point a in $K3$ support the gauge group $Sp(2N_a)$ and have the Chern-Simons

couplings [4]

$$S_{H5}^{SO(32)} = -\frac{2\pi N_a}{\ell_s^6}\int_{\mathbb{R}^{1,5}} B^{(6)} + \frac{1}{4\pi\ell_s^2}\int_{\mathbb{R}_{1,5}} B^{(2)}\wedge\left(\frac{N_a}{24}\mathrm{tr}R^2 - \mathrm{tr}_{Sp(2N_a)}F^2\right) \quad (20)$$

which were derived by S-duality [27] arguments and are consistent with four-dimensional anomaly cancellation.

Introducing the skyscraper sheafs $\mathcal{O}|_a$ with

$$\mathrm{ch}\,(\mathcal{O}|_a) = (0,0,-1)$$

allows to generalise the index (2) to extensions Ext^*_{K3}. The massless spectrum is thus computed along the same lines as for the four dimensional case in [3, 4] by decomposing the adjoint of $SO(32)$ and identifying the bundles. The complete six-dimensional massless spectrum is listed in table 2.[8]

reps.	# Hyper	# Vector
$(\mathbf{Adj}_{U(N_i)})_{0(i)}$	$1-\frac{1}{2}\chi(V_i\otimes V_i^*)$	1
$(\mathbf{Sym}_{U(N_i)})_{2(i)}+c.c.$	$-\chi(\wedge^2 V_i)$	0
$(\mathbf{Anti}_{U(N_i)})_{2(i)}+c.c.$	$-\chi(\otimes_s^2 V_i)$	0
$(\mathbf{N}_i,\mathbf{N}_j)_{1(i),1(j)}+c.c.$	$-\chi(V_i\otimes V_j)$	0
$(\mathbf{N}_i,\overline{\mathbf{N}}_j)_{1(i),-1(j)}+c.c.$	$-\chi(V_i\otimes V_j^*)$	0
$(\mathbf{Adj}_{SO(2M)})$	0	1
$(2\mathbf{M},\mathbf{N}_i)_{1(i)}+c.c.$	-$\chi(V_i)$	0
$(\mathbf{Sym}_{Sp(2N_a)})$	0	1
$(\mathbf{Anti}_{Sp(2N_a)})$	1	0
$(\mathbf{N}_i,2\mathbf{N}_a)_{1(i)}+c.c.$	n_i	0
$(2\mathbf{M},2\mathbf{N}_a)$	$\frac{1}{2}$	0

Table 2: Six-dimensional charged spectrum for the $SO(32)$ heterotic string with bundles of type (19) and H5-branes. The spectrum is completed by the supergravity and one universal tensor multiplet as well as 20 neutral hyper multiplets encoding the $K3$ geometry.

The tadpole cancellation condition for bundles of type (19) is on $K3$ given by

$$\sum_{i=1}^{K} N_i\,\mathrm{ch}_2(V_i) - \sum_{a=1}^{L} N_a = -c_2(T) = -24, \quad (21)$$

[8]Note that although $\chi(\mathcal{O}|_a,\mathcal{O}|_a)=0$, there exists a vector multiplet in the symmetric(=adjoint) and a hyper multiplet in the antisymmetric representation of $Sp(2N_a)$ in the spectrum.

where $\sum_{a=1}^{L} N_a = N_{H5}$ is the total number of H5-branes.

In total, we have a vanishing $\mathrm{tr}R^4$ anomaly due to

$$\begin{aligned}
n_T &= 1, \\
n_H - n_V &= 20 - \sum_i \frac{N_i}{2}\left(N_i\chi(V_i\otimes V_i^*) + (N_i+1)\chi(\wedge^2 V_i) + (N_i-1)\chi(\otimes_s^2 V_i)\right) \\
&\quad -\sum_{i<j} N_i N_j \left(\chi(V_i\otimes V_j) + \chi(V_i\otimes V_j^*)\right) - M(2M-1) \\
&\quad -2M\sum_i N_i\chi(V_i) - N_a(2N_a+1) + N_a(2N_a-1) + 2N_a(\sum_i n_i N_i + M) \\
&= 244, \qquad (22)
\end{aligned}$$

where in the last line the tadpole cancellation condition (21) and the properties of Chern characters, e.g. $\mathrm{ch}(V\otimes W) = \mathrm{ch}(V)\cdot\mathrm{ch}(W)$, have been used.[9]

This leads together with table 2 to the anomaly eight-form

$$\begin{aligned}
I_8^{SO(32)} &= \left(\mathrm{tr}R^2\right)^2 + \mathrm{tr}R^2\left(\mathrm{tr}_{SO(2M)}F^2 - 2\,\mathrm{tr}_{Sp(2N_a)}F^2\right) \\
&\quad +\mathrm{tr}R^2\left(\sum_i 2\left(n_i + 2\mathrm{ch}_2(V_i)\right)\mathrm{tr}_{U(N_i)}F^2 + \frac{1}{3}\left(\sum_i c_1(V_i)\,\mathrm{tr}_{U(N_i)}F\right)^2\right) \\
&\quad -\frac{16}{3}\sum_{i,j} c_1(V_i)c_1(V_j)\,\mathrm{tr}_{U(N_i)}F\,\mathrm{tr}_{U(N_j)}F^3 \\
&\quad -2\left(\mathrm{tr}_{SO(2M)}F^2\right)^2 - 8\left(\sum_i n_i \mathrm{tr}_{U(N_i)}F^2\right)\left(\sum_j (\mathrm{ch}_2(V_j)+n_j)\mathrm{tr}_{U(N_j)}F^2\right) \\
&\quad +2\,\mathrm{tr}_{Sp(2N_a)}F^2\,\mathrm{tr}_{SO(2M)}F^2 + 4\,\mathrm{tr}_{Sp(2N_a)}F^2\sum_i n_i\mathrm{tr}_{U(N_i)}F^2 \\
&\quad -4\,\mathrm{tr}_{SO(2M)}F^2\sum_i \mathrm{tr}_{U(N_i)}F^2\left(\mathrm{ch}_2(V_i) + 2n_i\right),
\end{aligned}$$

which can be rewritten in the partially factorised form

$$\begin{aligned}
I_8^{SO(32)} &= \left(\mathrm{tr}R^2 - \mathrm{tr}_{SO(2M)}F^2 - 2\sum_i n_i\,\mathrm{tr}_{U(N_i)}F^2\right)\times \qquad (23) \\
&\quad \times\left(\mathrm{tr}R^2 + 2\,\mathrm{tr}_{SO(2M)}F^2 + 4\sum_j(\mathrm{ch}_2(V_j)+n_j)\mathrm{tr}_{U(N_j)}F^2 - 2\,\mathrm{tr}_{Sp(2N_a)}F^2\right) \\
&\quad +\frac{1}{3}\left(\sum_i c_1(V_i)\,\mathrm{tr}_{U(N_i)}F\right)\times\left(\sum_j c_1(V_j)\left[\mathrm{tr}R^2\,\mathrm{tr}_{U(N_j)}F - 16\,\mathrm{tr}_{U(N_j)}F^3\right]\right),
\end{aligned}$$

[9]The index (2) on $K3$ contains only *even* Chern characters in contrast to CY_3 compactifications where only *odd* ones appear. The relations among the first Chern characters of $\wedge^2 V$ and the rank r bundle V are e.g. listed in eq. (19) in [2], and by using the relation $V\otimes V = (\wedge^2 V)\oplus(\otimes_s^2 V)$ one obtains $\mathrm{ch}(\otimes_s^2 V) = \frac{r(r+1)}{2} + (r+1)c_1(V) + \left[(r+2)\mathrm{ch}_2(V) + \frac{1}{2}c_1(V)^2\right] + \cdots$.

whose non-Abelian part is in agreement e.g. with the special cases $SO(32) \times Sp(48)$ [15] and $SO(28) \times SU(2)$ for $(N, n, \text{ch}_2(V)) = (2, 1, -12)$ [16,21].

It can be checked explicitly that all non-Abelian $\text{tr}F^4$ anomalies vanish upon tadpole cancellation, e.g.

$$\begin{aligned} I_{\text{tr}_{U(N_i)}F^4} \;\sim\;& N_i \chi(V_i \otimes V_i^*) + (N_i + 8)\chi(\wedge^2 V_i) + (N_i - 8)\chi(\otimes_s^2 V_i) \\ &+ \sum_{j \neq i} N_j \left(\chi(V_i \otimes V_j) + \chi(V_i \otimes V_j^*)\right) + 2M\chi(V_i) + 2N_a n_i = 0, \end{aligned}$$

for the spectrum in table 2.

In the following section we show that the Green-Schwarz counter-terms have exactly the correct shape to cancel all the anomalies encoded in the polynomial (23).

3.2 Anomaly cancellation for $SO(32)$

Inserting the expansion (15) in the H5-brane Chern-Simons action gives

$$S_{H5}^{SO(32)} = -\frac{2\pi N_a}{\ell_s^6} \int_{\mathbb{R}^{1,5}} c_0^{(6)} - \frac{1}{4\pi \ell_s^2} \int_{\mathbb{R}_{1,5}} b_0^{(2)} \wedge \left(\text{tr}_{Sp(2N_a)} F^2 - \frac{N_a}{24} \text{tr} R^2\right) \quad (24)$$

which provides the missing term for the tadpole cancellation condition and leads to the non-perturbative part of the Green-Schwarz counter-terms,

$$\mathcal{I}_{non-pert} = \frac{1}{192(2\pi)^2 \ell_s^4} \left(\text{tr}F^2 - \text{tr}R^2\right) \wedge \left(N_{H5} \text{tr}R^2 - 24 \sum_{a=1}^{L} \text{tr}_{Sp(2N_a)} F^2\right). \quad (25)$$

For the class of bundles (19), the relevant polynomials are given by

$$\begin{aligned} X_{\bar{4}+4} &= \sum_{j=1}^{K} \text{tr}_{U(N_j)} F^2 \left(12\, \text{tr}_{U(n_j)} \overline{F}^2 - \frac{n_j}{4} \text{tr} \overline{R}^2\right) - \frac{1}{8} \text{tr}_{SO(2M)} F^2\, \text{tr} \overline{R}^2 \\ &\quad + \text{tr}R^2 \left(\frac{1}{16} \text{tr} \overline{R}^2 - \frac{1}{4} \sum_{j=1}^{K} N_j \text{tr}_{U(n_j)} \overline{F}^2\right), \\ X_{\bar{2}+6} &= \sum_{j=1}^{K} \text{tr}_{U(n_j)} \overline{F} \left(8\, \text{tr}_{U(N_j)} F^3 - \frac{1}{2} \text{tr}_{U(N_j)} F \wedge \text{tr} R^2\right), \end{aligned} \quad (26)$$

and all other traces can be extracted from appendix B in [4].

From (17), (25) and (26), the complete Green-Schwarz counter-term can be computed,

$$\mathcal{I}_{pert} + \mathcal{I}_{non-pert} = -\frac{1}{96(2\pi \ell_s)^4} I_8^{SO(32)}. \quad (27)$$

As required, the counter-terms match (up to some normalisation constant) minus the anomaly eight-form (23).

The masses (18) of the Abelian gauge factors for the class of models presented here are given by

$$S^{SO(32)}_{mass} = \sum_{k=0}^{h_{11}+1} \sum_{i=1}^{K} \frac{M_i^k}{\ell_s^4} \int_{\mathbb{R}^{1,5}} c_k^{(4)} \wedge f_i \qquad \text{with} \qquad M_i^k = [N_i\, c_1(V_i)]^k \,,$$

where f_i is the $U(1)$ part of the $U(N_i)$ gauge factor. The number of massive Abelian gauge factors is given by rank(M).

3.3 Example 1: $U(3) \times U(3)$ bundle without H5-branes

An example on $\mathcal{M}_2$ with $U(3) \times U(3)$ bundles is given by

$$\begin{aligned} 0 \to V_1 \to \mathcal{O}(1,0)^{\oplus 2} \oplus \mathcal{O}(0,1)^{\oplus 2} \to \mathcal{O}(1,3) \to 0, \\ 0 \to V_2 \to \mathcal{O}(1,0)^{\oplus 2} \oplus \mathcal{O}(0,1)^{\oplus 2} \to \mathcal{O}(3,1) \to 0, \end{aligned}$$

from which the Chern characters are computed,

$$\begin{aligned} c_1(V_1) &= \eta_1 - \eta_2, \qquad \mathrm{ch}_2(V_1) = -16, \\ c_1(V_2) &= \eta_2 - \eta_1, \qquad \mathrm{ch}_2(V_2) = -8. \end{aligned}$$

In particular, both bundles have the same rank but different instanton numbers, i.e. $\mathrm{ch}_2(V_1) \neq \mathrm{ch}_2(V_2)$, due to the asymmetric shape of the intersection form (8) on $\mathcal{M}_2$.

The bundle

$$V = V_1 \oplus V_2$$

saturates the tadpole cancellation condition and satisfies the K-theory constraint trivially with $c_1(V) = 0$. The resulting spectrum is listed in table 3. Due to

$SO(20) \times U(1)^2$	# H	# V	$SO(20) \times U(1)^2$	# H
$(\mathbf{1})_{0,0}$	52	2	$(\mathbf{1})_{2,0} + c.c.$	12
$(\mathbf{190})_{0,0}$	0	1	$(\mathbf{1})_{0,2} + c.c.$	4
$(\mathbf{20})_{1,0} + c.c.$	10	0	$(\mathbf{1})_{1,1} + c.c.$	50
$(\mathbf{20})_{0,1} + c.c.$	2	0	$(\mathbf{1})_{1,-1} + c.c.$	58

Table 3: Charged spectrum of example 1 including massive $U(1)$ factors.

$c_1(V_1) = -c_1(V_2)$, the linear combination $U(1)_1 + U(1)_2$ remains massless, while

its orthogonal combination becomes massive by absorbing one neutral hyper multiplet.

The DUY condition is identical for both bundles,

$$\rho_2 = 3\rho_1$$

and freezes one Kähler modulus as expected from the existence of one massive vector. It can be easily fulfilled inside the Kähler cone.

3.4 Example 2: $U(3) \times U(3) \times U(1)$ bundle with H5-branes

As a second example, consider the following $U(3) \times U(3) \times U(1)$ bundle

$$V = V_1 \oplus V_2 \oplus L$$

on $\mathcal{M}_3$ defined by

$$\begin{aligned} &0 \to V_1 \to \mathcal{O}(1,0,0)^{\oplus 2} \oplus \mathcal{O}(0,1,0)^{\oplus 2} \to \mathcal{O}(2,1,1) \to 0, \\ &0 \to V_2 \to \mathcal{O}(0,0,1)^{\oplus 2} \oplus \mathcal{O}(0,1,0)^{\oplus 2} \to \mathcal{O}(1,2,1) \to 0, \end{aligned} \tag{28}$$

with the Chern characters

$$\begin{aligned} c_1(V_1) &= \eta_2 - \eta_3, \qquad \mathrm{ch}_2(V_1) = -10, \\ c_1(V_2) &= \eta_3 - \eta_1, \qquad \mathrm{ch}_2(V_2) = -10, \end{aligned}$$

as well as the line bundle

$$c_1(L) = \eta_1 - \eta_2, \qquad \mathrm{ch}_2(L) = -2.$$

The K-theory constraint is again trivially fulfilled with $c_1(V) = 0$. The DUY equations for V_1, V_2 and L require $\rho_2 = \rho_3$, $\rho_1 = \rho_3$ and $\rho_1 = \rho_2$, respectively.

In order to satisfy the tadpole cancellation condition, two H5-branes are needed. The charged spectrum for coincident H5-branes is listed in table 4. The combination $U(1)_1 + U(1)_2 + U(1)_3$ remains massless, while the two orthogonal linear combinations become massive by absorbing one hyper multiplet each. This agrees with the fact that $\rho_1 = \rho_2 = \rho_3$ freezes two Kähler moduli.

4 The $E_8 \times E_8$ case

4.1 A specific class of models

In order to check the general form of the counter-terms to be derived in sections 4.2 and 4.3, we introduce a series of decompositions of the form $E_8^{(i)} \to E_{r_i} \times SU(n_i + m_i)$ with $i = 1, 2$, $r_i + n_i + m_i = 9$, and $E_r = E_7, E_6, SO(10), SU(5), SU(2) \times SU(3)$

$SO(18)\times Sp(4)\times U(1)^3$	# H	# V	$SO(18)\times Sp(4)\times U(1)^3$	# H
$(\mathbf{1},\mathbf{1})_{0,0,0}$	40	3	$(\mathbf{1},\mathbf{4})_{1,0,0}+c.c.$	3
$(\mathbf{153},\mathbf{1})_{0,0,0}$	0	1	$(\mathbf{1},\mathbf{4})_{0,1,0}+c.c.$	3
$(\mathbf{1},\mathbf{10})_{0,0,0}$	0	1	$(\mathbf{1},\mathbf{4})_{0,0,1}+c.c.$	1
$(\mathbf{18},\mathbf{1})_{1,0,0}+c.c.$	4	0	$(\mathbf{18},\mathbf{4})_{0,0,0}$	$\frac{1}{2}$
$(\mathbf{18},\mathbf{1})_{0,1,0}+c.c.$	4	0	$(\mathbf{1},\mathbf{6})_{0,0,0}$	1
$(\mathbf{1},\mathbf{1})_{2,0,0}+c.c.$	6	0	$(\mathbf{1},\mathbf{1})_{1,0,1}+c.c.$	8
$(\mathbf{1},\mathbf{1})_{0,2,0}+c.c.$	6	0	$(\mathbf{1},\mathbf{1})_{1,0,-1}+c.c.$	12
$(\mathbf{1},\mathbf{1})_{1,1,0}+c.c.$	40	0	$(\mathbf{1},\mathbf{1})_{0,1,1}+c.c.$	8
$(\mathbf{1},\mathbf{1})_{1,-1,0}+c.c.$	44	0	$(\mathbf{1},\mathbf{1})_{0,1,-1}+c.c.$	12

Table 4: Charged spectrum of example 2 including massive $U(1)$ factors.

for $r=7,6,5,4,3$. Furthermore, we embed bundles with structure group $SU(n_i)\times SU(m_i)\times U(1)_i$ in $SU(n_i+m_i)$ by either using bundles of the form

$$\begin{aligned} V &= V_1\oplus V_2, \\ V_i &= V_{n_i}\oplus V_{m_i}\oplus L_i \quad \text{with} \quad c_1(V_{n_i})=c_1(V_{m_i})=0,\ c_1(L_i)\neq 0, \end{aligned} \tag{29}$$

or

$$\begin{aligned} W &= W_1\oplus W_2, \\ W_i &= W_{n_i}\oplus W_{m_i} \quad \text{with} \quad c_1(W_{n_i})=-c_1(W_{m_i})\neq 0. \end{aligned} \tag{30}$$

In the latter case, the total bundle in each E_8 factor has vanishing first Chern class, $c_1(W_i)=0$, and K-theory does not further constrain the bundles.

The spectrum is computed along the lines described in [2] and gives again the generalisation of [9] to non-Abelian bundles. For example, consider the decomposition $E_8\to SU(5)\times SU(5)\to SU(3)\times SU(2)\times U(1)\times SU(5)$ and the corresponding breaking $\mathbf{248}\to(\mathbf{10},\mathbf{5})+\ldots\to(\mathbf{3},\mathbf{2};\mathbf{5})_1+(\overline{\mathbf{3}},\mathbf{1};\mathbf{5})_{-4}+(\mathbf{1},\mathbf{1};\mathbf{5})_6+\ldots$ from which the bundles associated to the observable $\mathbf{5}$ representations with different $U(1)$ charges are read off as $W_3\otimes W_2$, $\wedge^2 W_3$ and $\wedge^2 W_2$, respectively.

The general resulting spectrum for bundles of type (30) is displayed in tables 5 and 6. The multiplicities for bundles of type (29) are obtained from the same tables by simply replacing

$$W_n=V_n\otimes L^{-m/\mu},\quad W_m=V_m\otimes L^{n/\mu}, \tag{31}$$

with $\mu\equiv\gcd(n,m)$.

The tadpole cancellation condition for bundles of type (29) reads

$$\sum_{i=1}^{2}\left[\mathrm{ch}_2(V_{n_i})+\mathrm{ch}_2(V_{m_i})+b_{n_i,m_i}\,\mathrm{ch}_2(L_i)\right]-N_{H5}=-c_2(T)=-24, \tag{32}$$

E_7	E_6	$SO(10)$	$SO(10)$	$SU(5)$	$SU(5)$	E_r	# H	# V
$(1,1)$	$(2,1)$	$(3,1)$	$(2,2)$	$(4,1)$	$(3,2)$	(n,m)		
$(\mathbf{133})_0$	$(\mathbf{78})_0$	$(\mathbf{45})_0$	$(\mathbf{45})_0$	$(\mathbf{24})_0$	$(\mathbf{24})_0$	$(\mathbf{Adj})_0$	0	1
$(\mathbf{1})_0$	$(\mathbf{1})_0$	$(\mathbf{1})_0$	$(\mathbf{1})_0$	$(\mathbf{1})_0$	$(\mathbf{1})_0$	$(\mathbf{1})_0$	$2-\frac{1}{2}\chi(W_n\otimes W_n^*)$ $-\frac{1}{2}\chi(W_m\otimes W_m^*)$	1
$(\mathbf{56})_{-1}^{+cc}$	$(\mathbf{27})_{-1}^{+cc}$	$(\mathbf{16})_{-1}^{+cc}$	$(\mathbf{16})_{-1}^{+cc}$	$(\overline{\mathbf{10}})_{-1}^{+cc}$	$(\overline{\mathbf{10}})_{-2}^{+cc}$	$(\mathbf{X})_{\frac{-m}{\mu}}^{+cc}$	$-\chi(W_n)$	0
-	$(\mathbf{27})_{2}^{+cc}$	$(\mathbf{16})_{3}^{+cc}$	$(\mathbf{16})_{1}^{+cc}$	$(\overline{\mathbf{10}})_{4}^{+cc}$	$(\overline{\mathbf{10}})_{3}^{+cc}$	$(\mathbf{X})_{\frac{n}{\mu}}^{+cc}$	$-\chi(W_m)$	0
-	-	$(\mathbf{10})_{2}^{+cc}$	$\frac{1}{2}(\mathbf{10})_0$	$(\mathbf{5})_{3}^{+cc}$	$(\mathbf{5})_{1}^{+cc}$	$(\mathbf{F})_{\frac{n-m}{\mu}}^{+cc}$	$-\chi(W_n\otimes W_m)$	0
-	-	-	$(\mathbf{10})_{2}^{+cc}$	$(\mathbf{5})_{-2}^{+cc}$	$(\mathbf{5})_{-4}^{+cc}$	$(\mathbf{F})_{\frac{-2m}{\mu}}^{+cc}$	$-\chi(\wedge^2 W_n)$	0
-	-	-	-	-	$(\mathbf{5})_{6}^{+cc}$	$(\mathbf{F})_{\frac{2n}{\mu}}^{+cc}$	$-\chi(\wedge^2 W_m)$	0
$(\mathbf{1})_{-2}^{+cc}$	$(\mathbf{1})_{-3}^{+cc}$	$(\mathbf{1})_{-4}^{+cc}$	$(\mathbf{1})_{-2}^{+cc}$	$(\mathbf{1})_{-5}^{+cc}$	$(\mathbf{1})_{-5}^{+cc}$	$(\mathbf{1})_{-\frac{n+m}{\mu}}^{+cc}$	$-\chi(W_n\otimes W_m^*)$	0

Table 5: Part I: Six-dimensional spectrum from the breaking of a single E_8 factor. The spectrum is completed by the states from the second E_8 factor, the supergravity and one universal tensor multiplet, 20 neutral hyper multiplets encoding the $K3$ geometry as well as one tensor and neutral hyper multiplet per H5-brane. For shortness we abbreviate $\mu \equiv \gcd(n,m)$ and label by $^{+cc}$ the complex conjugate representation.

with $b_{n,m} = \frac{nm(n+m)}{(\gcd(n,m))^2}$ and for bundles of type (30) it is given by

$$\sum_{i=1}^{2} \left[\mathrm{ch}_2(W_{n_i}) + \mathrm{ch}_2(W_{m_i})\right] - N_{H5} = -c_2(T) = -24. \tag{33}$$

The gravitational anomalies are determined by

$$n_T = 1 + N_{H5}, \quad n_H - n_V = 244 - 29\, N_{H5}, \tag{34}$$

for an arbitrary $E_8 \times E_8$ heterotic compactification to six dimensions. The resulting anomaly polynomial for bundles of type (30) reads

$$\begin{aligned} I_8^{E_8\times E_8} \;=\; & \left[1 - \frac{N_{H5}}{8}\right] \left(\mathrm{tr}R^2\right)^2 \\ & + \mathrm{tr}R^2 \sum_{i=1}^{2} \Big[\frac{a_{r_i}}{2} \mathrm{tr}_{E_{r_i}} F^2 \left(\mathrm{ch}_2(W_{n_i}) + \mathrm{ch}_2(W_{m_i}) + 10\right) \\ & \qquad + b_{n_i,m_i} f_i^2 \left(\mathrm{ch}_2(W_{n_i}) + \mathrm{ch}_2(W_{m_i}) + a_{n_i,m_i} c_1(W_{n_i})^2 + 10\right)\Big] \end{aligned}$$

			$SU(2)\times SU(3)$	# H	# V
$(5,1)$	$(4,2)$	$(3,3)$	(n,m)		
$(\mathbf{3},\mathbf{1})_0$	$(\mathbf{3},\mathbf{1})_0$	$(\mathbf{3},\mathbf{1})_0$	$(\mathbf{Adj}_2,\mathbf{1})_0$	0	1
$(\mathbf{1},\mathbf{8})_0$	$(\mathbf{1},\mathbf{8})_0$	$(\mathbf{1},\mathbf{8})_0$	$(\mathbf{1},\mathbf{Adj}_3)_0$	0	1
$(\mathbf{1},\mathbf{1})_0$	$(\mathbf{1},\mathbf{1})_0$	$(\mathbf{1},\mathbf{1})_0$	$(\mathbf{1},\mathbf{1})_0$	$2-\frac{1}{2}\chi(W_n\otimes W_n^*)$ $-\frac{1}{2}\chi(W_m\otimes W_m^*)$	1
$(\mathbf{2},\mathbf{3})_{-1}^{+cc}$	$(\mathbf{2},\mathbf{3})_{-1}^{+cc}$	$(\mathbf{2},\mathbf{3})_{-1}^{+cc}$	$(\mathbf{2},\mathbf{3})_{-\frac{m}{\mu}}^{+cc}$	$-\chi(W_n)$	0
$(\mathbf{2},\mathbf{3})_{5}^{+cc}$	$(\mathbf{2},\mathbf{3})_{2}^{+cc}$	$(\mathbf{2},\mathbf{3})_{1}^{+cc}$	$(\mathbf{2},\mathbf{3})_{\frac{n}{\mu}}^{+cc}$	$-\chi(W_m)$	0
$(\mathbf{1},\overline{\mathbf{3}})_{4}^{+cc}$	$(\mathbf{1},\overline{\mathbf{3}})_{1}^{+cc}$	$(\mathbf{1},\overline{\mathbf{3}})_{0}^{+cc}$	$(\mathbf{1},\overline{\mathbf{3}})_{\frac{n-m}{\mu}}^{+cc}$	$-\chi(W_n\otimes W_m)$	0
$(\mathbf{1},\overline{\mathbf{3}})_{-2}^{+cc}$	$(\mathbf{1},\overline{\mathbf{3}})_{-2}^{+cc}$	$(\mathbf{1},\overline{\mathbf{3}})_{-2}^{+cc}$	$(\mathbf{1},\overline{\mathbf{3}})_{-\frac{2m}{\mu}}^{+cc}$	$-\chi(\wedge^2 W_n)$	0
-	$(\mathbf{1},\overline{\mathbf{3}})_{4}^{+cc}$	$(\mathbf{1},\overline{\mathbf{3}})_{2}^{+cc}$	$(\mathbf{1},\overline{\mathbf{3}})_{\frac{2n}{\mu}}^{+cc}$	$-\chi(\wedge^2 W_m)$	0
$(\mathbf{1},\mathbf{1})_{-5}^{+cc}$	$(\mathbf{1},\mathbf{1})_{-3}^{+cc}$	$(\mathbf{1},\mathbf{1})_{-2}^{+cc}$	$(\mathbf{1},\mathbf{1})_{-\frac{n+m}{\mu}}^{+cc}$	$-\chi(W_n\otimes W_m^*)$	0
$(\mathbf{2},\mathbf{1})_{-3}^{+cc}$	$(\mathbf{2},\mathbf{1})_{-3}^{+cc}$	$(\mathbf{2},\mathbf{1})_{-3}^{+cc}$	$(\mathbf{2},\mathbf{1})_{-\frac{3m}{\mu}}^{+cc}$	$-\chi(\wedge^3 W_n)$	0
-	$\frac{1}{2}(\mathbf{2},\mathbf{1})_0$	$(\mathbf{2},\mathbf{1})_{-1}^{+cc}$	$(\mathbf{2},\mathbf{1})_{\frac{n-2m}{\mu}}^{+cc}$	$-\chi((\wedge^2 W_n)\otimes W_m)$	0

Table 6: Part II: Six-dimensional spectrum for the decomposition $E_8\to SU(2)\times SU(3)\times SU(6)$ and $U(n)\times U(m)$ bundles embedded in $SU(6)$.

$$
\begin{aligned}
-\sum_{i=1}^{2}\Big\{ & \frac{a_{r_i}^2}{2}\left(\mathrm{tr}_{E_{r_i}}F^2\right)^2\left(\mathrm{ch}_2(W_{n_i})+\mathrm{ch}_2(W_{m_i})+12\right)\\
& +2a_{r_i}b_{n_i,m_i}\,f_i^2\,\mathrm{tr}_{E_{r_i}}F^2\left(\mathrm{ch}_2(W_{n_i})+\mathrm{ch}_2(W_{m_i})+a_{n_i,m_i}c_1(W_n)^2+12\right)\\
& +2b_{n_i,m_i}^2\,f_i^4\left(\mathrm{ch}_2(W_{n_i})+\mathrm{ch}_2(W_{m_i})+2a_{n_i,m_i}c_1(W_{n_i})^2+12\right)\Big\}. \qquad (35)
\end{aligned}
$$

The coefficients are defined as follows,

$$
\begin{aligned}
a_r &= \frac{1}{6},\frac{1}{4},1,2,(2,2) \quad \text{for} \quad r=7,6,5,4,(2,1) \qquad (36)\\
a_{n,m} &= \frac{n+m}{n\,m}, \qquad b_{n,m}=\frac{n\,m\,(n+m)}{(\mathrm{gcd}(n,m))^2}, \qquad \kappa_{n,m}=\frac{n+m}{\mathrm{gcd}(n,m)},
\end{aligned}
$$

and the polynomial for bundles of type (29) is easily obtained by using relation (31).

4.2 The perturbative Green-Schwarz counter-terms

Before computing the counter-terms for the embeddings (29), (30) we start by rewriting X_8 for $E_8\times E_8$ in full generality. Its relevant components for compact-

ifications to six dimensions are given by

$$
\begin{aligned}
X_{\bar{4}+4} &= \left\{\mathrm{tr}F_1^2\left(\frac{1}{2}\mathrm{tr}\overline{F}_1^2-\frac{1}{4}\mathrm{tr}\overline{F}_2^2-\frac{1}{8}\mathrm{tr}\overline{R}^2\right)+\left[\mathrm{tr}(F_1\overline{F}_1)\right]^2+(1\leftrightarrow 2)\right\}-\mathrm{tr}(F_1\overline{F}_1)\mathrm{tr}(F_2\overline{F}_2)\\
&\quad+\mathrm{tr}R^2\left(\frac{1}{16}\mathrm{tr}\overline{R}^2-\frac{1}{8}\mathrm{tr}\overline{F}_1^2-\frac{1}{8}\mathrm{tr}\overline{F}_2^2\right),\\
X_{\bar{2}+6} &= \left\{\mathrm{tr}F_1^2\left(\mathrm{tr}(F_1\overline{F}_1)-\frac{1}{2}\,\mathrm{tr}(F_2\overline{F}_2)\right)+(1\leftrightarrow 2)\right\}-\frac{1}{4}\,\mathrm{tr}R^2\left(\mathrm{tr}(F_1\overline{F}_1)+\mathrm{tr}(F_2\overline{F}_2)\right),
\end{aligned}
$$

and after inserting the tadpole cancellation condition (5), we obtain

$$
\begin{aligned}
X_{\bar{4}+4} &= \left\{\frac{3}{4}\,\mathrm{tr}F_1^2\left(\mathrm{tr}\overline{F}_1^2-\frac{1}{2}\mathrm{tr}\overline{R}^2\right)+\left[\mathrm{tr}(F_1\overline{F}_1)\right]^2+(1\leftrightarrow 2)\right\}-\mathrm{tr}(F_1\overline{F}_1)\mathrm{tr}(F_2\overline{F}_2)\\
&\quad-\frac{1}{16}\mathrm{tr}R^2\mathrm{tr}\overline{R}^2-4\pi^2N_{H5}\left[\mathrm{tr}F_1^2+\mathrm{tr}F_2^2\right]-2\pi^2N_{H5}\mathrm{tr}R^2,
\end{aligned}
$$

which can serve as a guidance to the correct contributions from H5-branes to the generalised Green-Schwarz mechanism.

The perturbative contributions to the counter-terms for any gauge background in $E_8\times E_8$ are therefore given by

$$
\begin{aligned}
\mathcal{I}_{pert} &= \frac{1}{48(2\pi\ell_s)^4}\int_{K3}\Bigg(\frac{3}{2}\left[\mathrm{tr}F_1^2\left\{\mathrm{tr}F_1^2\left(\frac{1}{4}\mathrm{tr}\overline{F}_1^2-\frac{1}{8}\mathrm{tr}\overline{R}^2\right)+\left(\mathrm{tr}(F_1\overline{F}_1)\right)^2\right\}+(1\leftrightarrow 2)\right]\\
&\qquad-\frac{1}{4}\mathrm{tr}R^2\left[\mathrm{tr}F_1^2\left(\frac{3}{2}\mathrm{tr}\overline{F}_1^2-\frac{5}{8}\mathrm{tr}\overline{R}^2\right)+3\left(\mathrm{tr}(F_1\overline{F}_1)\right)^2+(1\leftrightarrow 2)\right]\\
&\qquad+\frac{1}{32}(\mathrm{tr}R^2)^2\mathrm{tr}\overline{R}^2\Bigg) \qquad (37)\\
&\quad+\frac{N_{H5}}{192(2\pi)^2\ell_s^4}\left[2(\mathrm{tr}F_1^2)(\mathrm{tr}F_2^2)-2(\mathrm{tr}F_1^2)^2-2(\mathrm{tr}F_2^2)^2+\mathrm{tr}R^2\left(\mathrm{tr}F_1^2+\mathrm{tr}F_2^2\right)+(\mathrm{tr}R^2)^2\right].
\end{aligned}
$$

The discussion of the H5-brane contributions is postponed to section 4.3.

We now proceed to the comparison with the anomaly eight-form for the embeddings presented in section 4.1. The relevant traces for the spectra in tables 5 and 6 are computed as

$$
\mathrm{tr}F_i^2 = a_{r_i}\mathrm{tr}_{E_{r_i}}F^2+2\,b_{n_i,m_i}f_i^2,
$$

for both kinds of bundles and for bundles of type (29) we have

$$
\begin{aligned}
\mathrm{tr}(F_i\overline{F}_i) &= 2\,b_{n_i,m_i}f_i\overline{f}_i,\\
\mathrm{tr}\overline{F}_i^2 &= 2\left(\mathrm{tr}_{SU(n_i)}\overline{F}^2+\mathrm{tr}_{SU(m_i)}\overline{F}^2+b_{n_i,m_i}\overline{f}_i^2\right),
\end{aligned}
$$

with $\overline{f}_i=2\pi\,c_1(L_i)$, whereas for the bundle type (30) one obtains

$$
\begin{aligned}
\mathrm{tr}(F_i\overline{F}_i) &= 2\,\kappa_{n_i,m_i}f_i\overline{f}_i,\\
\mathrm{tr}\overline{F}_i^2 &= 2\left(\mathrm{tr}_{U(n_i)}\overline{F}^2+\mathrm{tr}_{U(m_i)}\overline{F}^2\right),
\end{aligned}
$$

with $\overline{f}_i = 2\pi\, c_1(W_{m_i})$. The constants $a_r, b_{n,m}, \kappa_{n,m}$ have been defined in (36).

It can be checked that all perturbative counter-terms have the correct shape to cancel the anomalies for $N_{H5} = 0$.

The mass terms for Abelian gauge fields are

$$S^{E_8\times E_8}_{mass} \;=\; \sum_{k=0}^{h_{11}+1}\sum_{i=1}^{2}\frac{M_i^k}{\ell_s^4}\int_{\mathbb{R}^{1,5}} c_k^{(4)}\wedge f_i$$

with

$$M_i^k = [b_{n_i,m_i}\, c_1(L_i)]^k\,, \qquad M_i^k = [\kappa_{n_i,m_i}\, c_1(W_{m_i})]^k\,,$$

for bundles of type (29) and (30), respectively. The number of massive vectors is given by rank(M).

4.3 H5-brane contributions to the Green-Schwarz mechanism

Little is known about the field theory of H5-branes in (compactifications of) the ten-dimensional $E_8\times E_8$ theory.[10] At this point, we use the anomaly polynomial to find the correct H5-brane contributions to the Green-Schwarz mechanism.

The purely gravitational eight form contribution is given in the first line of (3) in full generality, whereas all mixed and pure gauge anomalies depend on the specific embedding. However, to gather some information about possible H5-brane contributions, it is sufficient to notice that in no gauge anomaly computation from the spectrum the tadpole cancellation condition is used. Therefore, the overall counter-term for mixed and pure gauge anomalies must be independent of N_{H5}. With the knowledge of the perturbative part (37) and the counting of multiplets (34), this leads to the expected form

$$\mathcal{I}_{np} = \frac{N_{H5}}{192(2\pi)^2\ell_s^4}\left(2\,(\mathrm{tr}F_1^2)^2 + 2\,(\mathrm{tr}F_2^2)^2 - 2\,(\mathrm{tr}F_1^2)(\mathrm{tr}F_2^2) - \mathrm{tr}R^2\,(\mathrm{tr}F_1^2+\mathrm{tr}F_2^2) + \frac{1}{2}(\mathrm{tr}R^2)^2\right). \qquad (38)$$

In [14], it was noticed that the kinetic terms of the six-dimensional tensor fields contribute to the generalised Green-Schwarz mechanism. In the present discussion, these are exactly the anti-selfdual tensors with support on the H5-branes besides the universal tensor multiplet already taken into account in the perturbative counter-terms in section 4.2.

With the ansatz of one anti-selfdual field $d\tilde{b}_s^{(2)} = -\star_6\, d\tilde{b}_s^{(2)}$ per tensor multiplet $(s = 1,\ldots,N_{H5})$, the corresponding field strength takes the form

$$\tilde{H}_s^{(3)} \;=\; d\tilde{b}_s^{(2)} - \frac{\alpha'}{8}\left(a_1\,\omega_{Y,1} + a_2\,\omega_{Y,2} - b\,\omega_L\right) \qquad (39)$$

[10] There are some results starting from the M-theory picture, see e.g. [28].

where the constants a_i, b are not yet specified. If we take the kinetic term of the H5-brane tensor multiplet with the same normalisation as the one for the universal tensor multiplet used in section 4.2,

$$S_{kin} = -\frac{\pi}{\ell_s^8}\int_{\mathbb{R}^{1,5}\times K3} dB^{(2)}\wedge dB^{(6)} = -\frac{\pi}{\ell_s^4}\int_{\mathbb{R}^{1,5}} db_0^{(2)}\wedge dc_0^{(2)} + \cdots$$

we obtain[11]

$$\begin{aligned} S^{E_8\times E_8}_{kin,H5} &= -\frac{\pi}{\ell_s^4}\sum_{s=1}^{N_{H5}}\int_{\mathbb{R}^{1,5}} \tilde{H}_s^{(3)}\wedge\star_6\tilde{H}_s^{(3)} \qquad (40)\\ &= -\frac{\pi}{\ell_s^4}\sum_{s=1}^{N_{H5}}\int_{\mathbb{R}^{1,5}} d\tilde{b}_s^{(2)}\wedge\star_6 d\tilde{b}_s^{(2)}\\ &\quad +\frac{1}{8(2\pi)\ell_s^2}\sum_{s=1}^{N_{H5}}\int_{\mathbb{R}^{1,5}}\left(-a_1\,\mathrm{tr}F_1^2 - a_2\,\mathrm{tr}F_2^2 + b\,\mathrm{tr}R^2\right)\wedge\ \tilde{b}_s^{(2)}, \end{aligned}$$

and the counter-terms from $\tilde{b}_s^{(2)}\sim\tilde{b}_s^{(2)}$ exchange sum up to

$$\mathcal{I}_{H5,1} = \frac{N_{H5}}{128(2\pi)^2\ell_s^4}\left(\left[a_1\mathrm{tr}F_1^2 + a_2\mathrm{tr}F_2^2\right]^2 - 2b\,\mathrm{tr}R^2\left(a_1\mathrm{tr}F_1^2 + a_2\mathrm{tr}F_2^2\right) + b^2(\mathrm{tr}R^2)^2\right). \qquad (41)$$

We make furthermore the ansatz for a Chern-Simons like coupling of the form[12]

$$\begin{aligned} S^{E_8\times E_8}_{CS,H5} &= -\frac{2\pi}{\ell_s^6}\sum_{s=1}^{N_{H5}}\int_{\mathbb{R}^{1,5}} B^{(6)} \qquad (42)\\ &\quad +\frac{1}{96\pi\ell_s^2}\sum_{s=1}^{N_{H5}}\int_{\mathbb{R}_{1,5}} B^{(2)}\wedge\left[\eta_0\mathrm{tr}R^2 + \eta_1\mathrm{tr}F_1^2 + \eta_2\mathrm{tr}F_2^2\right], \end{aligned}$$

where the first term enters the tadpole cancellation condition and the terms in the second line contribute to the generalised Green-Schwarz mechanism via counter-terms involving the exchange of $b_0^{(2)}\sim c_0^{(2)}$,

$$\begin{aligned} \mathcal{I}_{H5,2} &= \frac{N_{H5}}{192(2\pi)^2\ell_s^4}\Big[\eta_1(\mathrm{tr}F_1^2)^2 + \eta_2(\mathrm{tr}F_2^2)^2 + (\eta_1+\eta_2)(\mathrm{tr}F_1^2)(\mathrm{tr}F_2^2)\\ &\qquad + \mathrm{tr}R^2\left[(\eta_0-\eta_1)\mathrm{tr}F_1^2 + (\eta_0-\eta_2)\mathrm{tr}F_2^2\right] - \eta_0(\mathrm{tr}R^2)^2\Big]. \qquad (43) \end{aligned}$$

[11]The canonical string frame normalisation of the kinetic term might involve a factor of g_s^{-2}. Modifying the anti-selfduality relation $d\tilde{b}_s^{(2)} = -g_s^{-2}\star_6 d\tilde{b}_s^{(2)}$ accordingly does not change the resulting counter-term.

[12]The gravitational coupling in (42) was found in [29, 30] in the ten-dimensional set-up by reduction from M-theory.

Together, we obtain

$$\begin{aligned}\mathcal{I}_{H5} &= \mathcal{I}_{H5,1} + \mathcal{I}_{H5,2} \qquad (44)\\ &= \frac{N_{H5}}{192(2\pi)^2\ell_s^4}\Big[\sum_{i=1}^{2}\alpha_i(\mathrm{tr}F_i^2)^2 + \beta\,(\mathrm{tr}F_1^2)(\mathrm{tr}F_2^2) + \sum_{i=1}^{2}\gamma_i(\mathrm{tr}F_i^2)(\mathrm{tr}R^2) + \delta\,(\mathrm{tr}R^2)^2\Big]\end{aligned}$$

with

$$\begin{aligned}\alpha_i \equiv \frac{3}{2}a_i^2 + \eta_i &\stackrel{!}{=} 2, \qquad \beta \equiv 3a_1a_2 + \eta_1 + \eta_2 \stackrel{!}{=} -2,\\ \gamma_i \equiv -3ba_i + \eta_0 - \eta_i &\stackrel{!}{=} -1, \qquad \delta \equiv \frac{3}{2}b^2 - \eta_0 \stackrel{!}{=} \frac{1}{2}. \qquad (45)\end{aligned}$$

In particular, we have the relations

$$(b-a_i)^2 \stackrel{!}{=} 1, \qquad (a_1-a_2)^2 \stackrel{!}{=} 4,$$

showing that $a_1 \neq a_2$ is necessary.

The overall counter-term takes the correct form for the most symmetric choice

$$a_1 = -1, \quad a_2 = 1, \quad b = 0, \quad \eta_0 = -\frac{1}{2}, \quad \eta_1 = \eta_2 = \frac{1}{2}. \qquad (46)$$

The non-perturbative Green-Schwarz counter-term (38) is thus induced by the following terms in the H5-brane action,

$$\begin{aligned}S_{kin,H5}^{E_8\times E_8} &= -\frac{\pi}{\ell_s^4}\sum_{s=1}^{N_{H5}}\int_{\mathbb{R}^{1,5}}\tilde{H}_s^{(3)}\wedge\star_6\tilde{H}_s^{(3)},\\ S_{CS,H5}^{E_8\times E_8} &= -\frac{2\pi}{\ell_s^6}\sum_{s=1}^{N_{H5}}\int_{\mathbb{R}^{1,5}}B^{(6)} \qquad (47)\\ &\quad +\frac{1}{192\pi\ell_s^2}\sum_{s=1}^{N_{H5}}\int_{\mathbb{R}_{1,5}}B^{(2)}\wedge\left[\mathrm{tr}F_1^2 + \mathrm{tr}F_2^2 - \mathrm{tr}R^2\right],\end{aligned}$$

with

$$\tilde{H}_s^{(3)} = d\tilde{b}_s^{(2)} + \frac{\alpha'}{8}\left(\omega_{Y,1} - \omega_{Y,2}\right). \qquad (48)$$

This result agrees with the computation of four-dimensional heterotic gauge anomalies in the presence of H5-branes [31].

4.4 Examples with $U(3)\times U(1)$ bundles

As an illustration, we present three models with bundles of type (30) which trivially fulfill the K-theory constraint, one with H5-branes and two without.

Consider one of the $U(3)$ bundles V_i defined in (28) and the corresponding line bundle L_i with $c_1(V_i) = -c_1(L_i)$ embedded in an E_8 factor. The resulting charged spectrum from this E_8 factor is given in table 7, and the corresponding DUY condition reads

$$\rho_2 = \rho_3 \quad \text{for} \quad V_1, \qquad \text{or} \qquad \rho_1 = \rho_3 \quad \text{for} \quad V_2. \tag{49}$$

There are three different obvious ways to satisfy the tadpole cancellation condi-

$SO(10) \times U(1)$	# H	# V	$SO(10) \times U(1)$	# H
$(\mathbf{45})_0$	0	1	$(\mathbf{16})_{-1} + c.c.$	4
$(\mathbf{1})_0$	20	1	$(\mathbf{10})_2 + c.c.$	6
			$(\mathbf{1})_{-4} + c.c.$	14

Table 7: Charged spectrum from embedding $U(3) \times U(1)$ in one E_8 factor. The $U(1)$ factor is in general massive.

tions:

1. The total bundle

$$V = V_i \oplus L_i, \qquad i = 1 \text{ or } 2$$

is embedded in one E_8 factor, the resulting gauge group is $SO(10) \times U(1) \times E_8'$, and twelve H5-branes are needed in order to fulfill the tadpole cancellation condition. The low-energy spectrum consists of a copy of the states in table 7, the vector in the adjoint of E_8', twelve tensor and hyper multiplets from the H5-branes and the universally present tensor, twenty neutral hyper and the supergravity multiplet. The Abelian gauge factor becomes massive. Fittingly the DUY condition gives one constraint on the Kähler moduli.

2. The total bundle is

$$V = (V_1 \oplus L_1) \oplus (V_2 \oplus L_2)$$

and the resulting gauge group $[SO(10) \times U(1)] \times [SO(10)' \times U(1)']$. The tadpole cancellation condition is satisfied without H5-branes. $c_1(V_1)$ and $c_1(V_2)$ are linearly independent leading to two massive Abelian gauge factors. Compatible with this fact, the DUY conditions freeze two Kähler moduli.

3. The total bundle contains two copies of the same vector bundle,

$$V = (V_i \oplus L_i)^{\oplus 2}, \qquad i = 1 \text{ or } 2,$$

resulting again in the gauge group $[SO(10)\times U(1)]\times[SO(10)'\times U(1)']$ and tadpole cancellation without H5-branes. In this case, the linear combination $U(1)-U(1)'$ remains massless while the orthogonal combination acquires a mass. As expected, the DUY condition freezes only one Kähler modulus.

By comparing examples 2 and 3, it is obvious that the second Chern characters are not sufficient to describe bundles. $\mathrm{ch}_2(V_1)=\mathrm{ch}_2(V_2)=-10$ are identical in these examples, but $c_1(V_1)\neq\pm c_1(V_2)$ determines the number of massive Abelian gauge factors.

5 Conclusions

In this article, the six-dimensional generalized Green-Schwarz mechanism for $K3$ compactifications of the heterotic $SO(32)$ and $E_8\times E_8$ string with arbitrary Abelian and non-Abelian bundles and five-branes has been derived. General classes of embeddings have been introduced and their anomaly-eight forms computed. The dimensional reduction of the ten-dimensional tree-level and one-loop counter-terms matches these anomaly eight-forms in the absence of H5-branes. For the $SO(32)$ string, the Chern-Simons couplings of H5-branes introduced in [4] serve to cancel all remaining six-dimensional field theory anomalies. For the H5-brane action in the $E_8\times E_8$ heterotic theory, the kinetic terms of the additional tensor multiplets (40) together with some Chern-Simons-like coupling (42) provide the correct Green-Schwarz counter-terms.

In contrast to the four-dimensional case, the six-dimensional theory admits two different types of Green-Schwarz diagrams depicted in figure 1: tensors are needed to cancel anomalies involving only non-Abelian gauge fields and gravity, while for Abelian gauge fields also four-forms and their scalar duals contribute to the anomaly cancellation. The linear couplings (18) to the four-forms render Abelian gauge fields massive, and the corresponding Donaldson-Uhlenbeck-Yau and holomorphicity conditions freeze three geometric moduli from the same hyper multiplet. The $U(1)$ masses depend on the first Chern classes of the respective bundles times some combinatorial factors specifying the embedding in $SO(32)$ or E_8.

The six-dimensional results are in full agreement with the four-dimensional observations on multiple anomalous $U(1)$ factors in heterotic compactifications apart from the difference that the DUY condition (most likely) does not receive any loop corrections in the present case as argued in section 2.1.

The classification of E_8 breakings with one $U(1)$ gauge factor in section 4.1 as well as more general products of several $U(n)$ bundles in one E_8 factor as in [2] give rise to a very large class of $E_8\times E_8$ string vacua with multiple $U(1)$ factors. Together with the class of $SO(32)$ models in section 3.1, this leads to many models beyond the classification of six dimensional $\mathcal{N}=1$ supergravity theories in [32].

The results are also relevant in order to understand better the F-theory and S-duality relations with orientifold compactifications to six dimensions, for which the analogous dimensional field theory reduction to six dimensions could be performed along the lines of [33].

It might be interesting to compactify the heterotic vacua presented here on an additional two-torus, and investigate further in the four-dimensional $\mathcal{N} = 2$ field theory set-up T-duality among the two heterotic theories as well as the relation to type II Calabi-Yau compactifications.

Finally, the toy examples presented in this article contain only up to three $(1,1)$-forms. Fully fledged models involving all 22 two-forms could be obtained via the spectral cover construction on elliptically fibered $K3$ manifolds [34, 35].

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